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6 STABLE MODEL REFERENCE ADAPTIVE CONTROL

IN THE PRESENCE OF BOUNDED DISTURBANCES

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Stable Model Reference Adaptive Control

in the Presence of Bounded Disturbances

Gerhard Kreisselmeier* and Kumpati S. Narendra

Abstract

The adaptive control of a linear time-invariant plant in the presence of bounded disturbances is considered. In addition to the usual assumptions made regarding the plant transfer function, it is also assumed that the high frequency gain k_p of the plant and an upper bound on the magnitude of the controller parameters, are known. *Under these conditions* the adaptive controller suggested assures the boundedness of all signals in the overall system.

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1. Introduction: A major step in the development of adaptive systems theory was the establishment in recent years of the global stability of several equivalent adaptive schemes [1-3]. When the sign of the high frequency gain k_p, the relative degree n^* and the order n of the transfer function of a plant with zeros in the left half plane are given it was shown that these schemes can be used to adaptively control the given plant in a globally stable fashion. The latter implies that the parameters and signals of the plant and the controller are bounded while the error between the plant output and the output of a reference model tends to zero. The question naturally arises as to how well these schemes perform in the presence of external disturbances. In a recent report [4] it was shown that when bounded output disturbances are present the parameter error vector $\Phi(t)$ can grow without bound even though the state error vector between plant and model is bounded. This can be attributed to the fact that the overall nonlinear adaptive system is only uniformly stable but not uniformly asymptotically stable. Hence it became evident that modifications in the basic adaptive schemes are necessary when bounded external disturbances are present.

One such modification suggested in [4] resulted from the observation that the adjustment of the parameter error vector (given by $\dot{\phi}(t)$) is known to be in the "right" direction only when the output error $e(t)$ is large. Hence, it was argued, if a bound on the disturbance is known and ϕ is adjusted only when the output error exceeds a computed threshold, the signals and the parameters of the system would be bounded. Such an adjustment corresponds to an adaptive law with a deadzone and in [4] it is shown that the above conclusions are indeed true. The presence of the deadzone however results in finite parameter and output errors even when no external disturbance is present.

In this paper an alternate approach is taken which retains the potential of obtaining zero output error $e(t)$ (and zero parameter error $\phi(t)$ if the input is rich) in the limit when no external disturbance is present. Assuming that the desired constant controller parameter vector θ^* has a norm less than a known upper bound, the adaptive law is suitably modified when $\|\theta(t)\|$ exceeds this bound. This, in turn, assures at the outset the boundedness of the parameter error vector $\phi(t)$ despite the presence of the external disturbance. Demonstrating that all the other signals within the adaptive control loop also remain bounded is the principal result of the paper.

Surprisingly enough, the difficulty in the proof of stability does not arise when some of the state variables grow rapidly (e.g. exponentially) but rather when they grow slowly with time. In addition $\|x(t)\|$ (where x is the state of the overall system) need not grow monotonically, so that the usual limiting arguments cannot be used directly. If $\|x(t)\|$ is assumed to grow without bound, the existence of an arbitrarily large interval $[t_1,t_2]$ over which $\|x(t)\|$ is large, can be established. It is shown that in such a case, the effect of the disturbance is relatively small and hence arguments similar to those in the disturbance free case apply. This, in turn, leads to a proof by contradiction.

2. Structure of the Adaptive Control System:

Let the plant to be controlled be represented by the equations

$$\begin{aligned} \dot{x}_p &= A_p x_p + b_p u_p + d_p \nu_1 \\ y_p &= c_p^T x_p + \nu_2 \end{aligned} \qquad (1)^*$$

where x_p, u_p and y_p are the state, input and output respectively and ν_1 and ν_2 are plant and output disturbances. The transfer function of the plant is given by

$$c_p^T(sI-A_p)^{-1}b_p = k_p \frac{N_p(s)}{D_p(s)} \triangleq W_p(s).$$

The following assumptions are made regarding $W_p(s)$ and the disturbances ν_1 and ν_2:

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* Throughout this report while representing a function of time the argument 't' is omitted for the sake of simplicity of notation when no confusion can arise.

(i) $N_p(s), D_p(s)$ are monic polynomials of degrees m and n respectively, and $\dim(x_p) = n$.

(ii) m and n are known but the coefficients of $N_p(s)$ and $D_p(s)$ are unknown.

(iii) $N_p(s)$ is a strictly stable polynomial.

(iv) k_p is known.

(v) ν_1, ν_2 are piecewise continuous and uniformly bounded time functions defined for all $t \in R^+$.

A reference model is defined by the equations

$$\begin{aligned} \dot{x}_m &= A_m x_m + b_m r \\ y_m &= c_m^T x_m \end{aligned} \tag{2}$$

and its transfer function is defined as

$$c_m^T (sI - A_m)^{-1} b_m = k_m \frac{1}{D_m(s)} \triangleq \frac{k_m}{k_p} W_m(s) .$$

It is assumed that:

(i) $D_m(s)$ is a monic, strictly stable polynomial.

(ii) the degree of $D_m(s)$ is $n^* \triangleq n-m$, and $\dim(x_m) = n^*$

and (iii) r is a piecewise continuous, uniformly bounded reference signal.

The objective is to control the plant in such a fashion that the output error between the plant and the model, i.e. $e_1 \triangleq y_p - y_m$, as well as all the state variables remain uniformly bounded. This objective applies both to the case of model following ($r \not\equiv 0$) as well as state regulation ($r \equiv 0, y_m \equiv 0$).

To meet the control objective, a controller described by the equations

$$\dot{v}_1 = F v_1 + g u_p \tag{3a}$$

$$\dot{v}_2 = F v_2 + g y_p \tag{3b}$$

$$u_p = v^T \theta + \frac{k_m}{k_p} r \tag{4}$$

is set up, where

$v^T \triangleq [v_1^T, v_2^T]$ and

(i) $\dim(v_1) = \dim(v_2) = n$.

(ii) F is an arbitrary, strictly stable matrix

(iii) (F,g) is a completely controllable pair

and θ is a vector of controller parameters to be adapted.

The equations for the adaptive scheme are:

$$\dot{\zeta}_1 = A_m \zeta_1 + b_m u_p \tag{5a}$$

$$\dot{\zeta}_2 = A_m \zeta_2 + b_m y_p \tag{5b}$$

$$\dot{\omega}_1 = F\omega_1 + g c_m^T \zeta_1 \tag{6a}$$

$$\dot{\omega}_2 = F\omega_2 + g c_m^T \zeta_2 \tag{6b}$$

$$\dot{\theta} = -\Gamma \frac{\omega[\omega^T\theta - c_m^T\zeta + y_p]}{1 + x^T x} - \Gamma\theta f(\theta) \tag{8}$$

$$f(\theta) = \begin{cases} (1 - \|\theta\|/\|\theta^*\|_{max})^2 & \text{if } \|\theta\| \geq \|\theta^*\|_{max} \\ 0 & \text{elsewhere} \end{cases} \tag{9}$$

where $x^T \triangleq [v^T, \zeta^T, \omega^T]$, $\zeta^T \triangleq [\zeta_1^T, \zeta_2^T]$, $\omega^T \triangleq [\omega_1^T, \omega_2^T]$ and

(i) $\Gamma = \Gamma^T > 0$ is an arbitrary gain matrix

(ii) $\|\theta^*\|_{max}$ is a known upper bound on the norm of the (unknown) matching controller parameter vector θ^*.

Since v is equal to the state of a nonminimal representation of the plant (except for exponentially decaying initial conditions and the bounded effect of ν_1)[1], the controller structure allows the generation of arbitrary state feedback. Therefore a parameter vector θ^* exists such that if $\theta(t) \equiv \theta^*$ then the poles of the closed loop system are the eigenvalues of F and the zeros of $D_m(s)N_p(s)$. Therefore the transfer function between the reference input r and the output of the plant y_p becomes $k_m W_m(s)$, as desired.

The block diagram of the control loop together with the disturbances ν_1 and ν_2 is shown in Figure 1a. If $\theta = \theta^* + \phi$, where ϕ denotes the parameter error vector, Figure 1a can be transformed as shown in Figures 1b-1e to yield an error model with a bounded disturbance ν at the output. From Figures 1f-1h it follows that

$$\omega(t)^T\theta^* - c_m^T \zeta_1(t) + y_p(t) - \nu(t) \equiv 0 , \tag{10}$$

ν being a bounded disturbance due to ν_1,ν_2 and initial condition transients.

Defining $\xi^T \triangleq [x_p^T, x^T]$ and using (10) in (8) the overall adaptive control system may be rewritten in the form

$$\dot{\xi} = A\xi + b\phi^T(t)v(t) + \mu(t) \tag{11}$$

$$\dot{\phi} = -\Gamma \frac{\omega(t)[\omega^T(t)\phi + \nu(t)]}{1 + x^T(t)x(t)} - \Gamma\theta f(\theta) \tag{12}$$

where A is a strictly stable matrix and $\mu(t)$ is a uniformly bounded input vector due to ν_1,ν_2 and r. The objective then is to show the global stability of the overall system (11)-(12).

3. Preliminary Analysis:

a) Adaptive Law:

Let $V \triangleq \frac{1}{2}\phi^T\Gamma^{-1}\phi$. Evaluating its time derivative along the trajectory of (12) gives

$$\dot{V} = -\frac{\phi^T\omega[\omega^T\phi + \nu]}{1 + x^T x} - \phi^T\theta f(\theta). \tag{13}$$

From

$$\phi^T\theta = (\theta-\theta^*)^T\theta \geq \theta^T\theta - \|\theta^*\|\cdot\|\theta\| \tag{14}$$

and the definition of $f(\theta)$ it follows that $\phi^T\theta f(\theta) \geq 0$. Moreover, $\phi^T\theta f(\theta) \sim \|\phi\|^4$ for $\|\phi\| >> \|\theta^*\|_{max}$. Since $\omega/(1 + x^T x)^{1/2}$ is uniformly bounded, this implies that $\dot{V}$ is negative definite for all $V \geq V'$ and some $V' < \infty$. Therefore V is uniformly bounded and hence $\phi(t)$ is also uniformly bounded. Together with (11) this implies that $\xi(t)$ cannot grow faster than an exponential.

Taking the integral of (13) we obtain

$$\int_{t_1}^{t_2} \left\{ \frac{(\phi^T\omega)^2}{1+x^T x} + \phi^T\theta f(\theta) \right\} d\tau \leq V(t_1) - V(t_2) + \int_{t_1}^{t_2} \frac{|\phi^T\omega\nu|}{1+x^T x} dt$$

$$\leq \Delta_1 + \Delta_2 \cdot \int_{t_1}^{t_2} \frac{1}{(1+x^T x)^{1/2}} dt \qquad (15)$$

where Δ_1 and Δ_2 are appropriate positive constants.

In the absence of external disturbances the right hand side of (15) reduces to a constant. Setting $t_2 = \infty$ it can then be concluded that $\phi^T\omega/(1+x^T x)^{1/2} \to 0$ as $t \to \infty$, which in turn gives $\phi^T v/(1+x^T x)^{1/2} \to 0$ and $e \to 0$ as $t \to \infty$ [5].

In the present situation with disturbances, however, the integral on the right hand side of (15) may tend to infinity as $t_2 \to \infty$, even though $x \to \infty$ as $t \to \infty$. Therefore, the conclusions to be drawn from (15) have to be modified suitably. This is done in section 4.

b) Control Loop:

Let $W \triangleq \frac{1}{2} \xi^T P \xi$, where P is a positive definite matrix and satisfies the equation $A^T P + PA = -2I$. Evaluating $\dot{W}$ along the trajectory of (11) gives

$$\dot{W} = -\xi^T\xi + \xi^T Pb(\phi^T v + \mu)$$

$$\leq -\xi^T\xi\left[1 - \frac{|\xi^T Pb|}{(\xi^T\xi)^{1/2}}\left(\frac{|\phi^T v|}{(\xi^T\xi)^{1/2}} + \frac{\|\mu\|}{(\xi^T\xi)^{1/2}}\right)\right] \qquad (16)$$

Since $\|\mu\|$ and $|\xi^T Pb|/(\xi^T\xi)^{1/2}$ are uniformly bounded there exist positive constants ε_1, $a < \infty$ such that

$$\xi^T(t)\xi(t) \geq a^2 \tag{17a}$$

$$|\phi^T(t)v(t)|/\sqrt{\xi^T(t)\xi(t)} \leq 2\varepsilon_1 \tag{17b}$$

implies that the bracket term in (16) is greater than 1/2, i.e.

$$\dot{W}(t) \leq -\kappa W(t) \tag{18}$$

where $\kappa = 1/2\ \lambda_{max}(P)$. Hence W decreases exponentially at least with rate κ whenever (17) holds. On the other hand, if $\xi^T(t)\xi(t) \geq 1$ then the term within brackets in (16) is bounded. Therefore a positive constant $\lambda < \infty$ exists such that

$$\dot{W}(t) \leq \lambda W(t). \tag{19}$$

Hence, whenever $\xi^T(t)\xi(t) \geq 1$, W can at most grow exponentially with rate λ.

In what follows we make use of the fact that v(t) is the internal state of a nonminimal representation of the plant, except for the bounded effect of the bounded disturbances. Therefore positive constants c_1, c_2 exist such that $\|x_p\| \leq (1+c_1)\|v\| + c_2$. Hence

$$\begin{aligned} \|x\| \leq \|\xi\| &\leq \|x_p\| + \|x\| \\ &\leq c_1\|x\| + c_2 \end{aligned} \tag{20}$$

In view of inequality (20) $\|\xi\|$ and $\|x\|$, when they are large, are equivalent as far as the arguments in the following section are concerned.

4. Stability of the Adaptive System:

Based on the description of the plant and the controller and the assumptions imposed on them as listed in section 2, we can now state our main result as follows:

Theorem:

The overall adaptive closed loop system described by equations (11) and (12) is globally stable in the sense that for arbitrary bounded initial conditions and bounded signal $\mu(t)$ (or equivalently $\nu_1(t), \nu_2(t)$ and r(t)), all states of the adaptive system remain uniformly bounded.

The basic idea of the proof of the theorem is as follows. The parameter error $\phi(t)$ has been made uniformly bounded at the outset by the introduction of the term $\Gamma\theta f(\theta)$ in the adaptive law (8) (see section 3). Therefore, only $x(t)$, (or equivalently $\xi(t)$) can grow without bound. If $x(t)$ grows without bound then, since it cannot grow faster than $\exp(\lambda t)$, it assumes values greater than a constant a over an interval of time of length a, where a is arbitrarily large. From (15) it follows that $(\phi^T\omega)^2/(1+x^Tx)$ and $\phi^T\theta f(\theta)$ become less than some ε and hence (Ref. Appendix) that $|\phi^T v|/(1+x^Tx)^{1/2}$ becomes less than ε_1 most of the time within this interval. The latter implies (equations 17b and 18) that $\|x(t)\|$ decreases at the rate $\exp(-\kappa t)$ most of the time, and can increase at most as $\exp(\lambda t)$ the rest of the time within the interval. Since the time interval over which $x(t)$ decreases is large compared to the interval on which it increases it follows that $\|x(t)\|$ will assume a value less than a on the interval, which contradicts the original assumption.

<u>Proof</u>:

Let us assume that

$$\lim_{t\to\infty} \sup_{\tau\le t} \|x(\tau)\| = \infty \ . \tag{21}$$

Then, since $\|x\|$ can grow at most exponentially there exist monotonically increasing sequences $\{t_i\},\{a_i\}$ with $\lim_{i\to\infty} t_i = \infty$, $\lim_{i\to\infty} a_i = \infty$ such that

$$\| x(t_i) \| = a_i \tag{22a}$$

$$\| x(t) \| \ge a_i \quad \text{for all } t \in [t_i, t_i + a_i]. \tag{22b}$$

Using (22b) in (15) we obtain

$$\int_{t_i}^{t_i+a_i} \left\{ \frac{(\phi^T\omega)^2}{(1+x^Tx)^{1/2}} + \phi^T\theta f(\theta) \right\} dt \le \Delta_1 + \Delta_2 \triangleq \Delta \ , \ (i \ge 0) \ . \tag{23}$$

Since $\dot{\phi}(t)$ and the time derivative of $\omega/(1+x^Tx)^{1/2}$ are uniformly bounded (see Appendix), there exists a constant C, such that

$$\left| \frac{d}{dt} \frac{(\phi^T \omega)^2}{1+x^T x} \right| \leq C \tag{24a}$$

and

$$\left| \frac{d}{dt} \{\phi^T \theta f(\theta)\} \right| \leq C \tag{24b}$$

Let $\varepsilon \in (0,C]$ be arbitrary and consider $i \geq i'$ such that $a_i \geq 2C\Delta/\varepsilon^3$. Then the interval $T_i \triangleq [t_i, t_i + a_i]$ can be expressed as the union of $N = 2C\Delta/\varepsilon^3$ disjoint subintervals, each being of length Δt_i greater than or equal to one.

If in any of these subintervals in T_i there exists an instant of time such that $(\phi^T \omega)^2/(1+x^T x) + \phi^T \theta f(\theta) \geq \varepsilon$ then (24) implies that this subinterval contributes to the integral on the left hand side of (23) an amount of at least $\varepsilon^2/2C$. Since the integral is bounded by Δ there can be at most $\Delta/(\varepsilon^2/2C) = 2C\Delta/\varepsilon^2$ of these subintervals. Let the set of all such subintervals be denoted by T_{i1}. The worst that can happen on T_{i1} is that W increases according to $\dot{W} = \lambda W$.

Let T_{i2} denote the set whose elements are the remaining $2C\Delta(1/\varepsilon^3 - 1/\varepsilon^2)$ subintervals so that $T_i = T_{i1} \cup T_{i2}$. We have $(\phi^T \omega)^2/(1+x^T x) + \phi^T \theta f(\theta) < \varepsilon$ when $t \in T_{i2}$. It is shown in the Appendix that this implies $|\phi^T v|/(1+x^T x)^{1/2} \leq h(\varepsilon)$, where $h(\cdot)$ is a continuous function with $\lim_{\varepsilon \to 0} h(\varepsilon) = 0$. Hence, if ε is sufficiently small, then we have $\dot{W} \leq -\kappa W$ on T_{i2}.

Therefore

$$W(t_i + (j+1)\Delta t_i) \leq W(t_i + j\Delta t_i) \cdot e^{\lambda_j \Delta t_i}, \quad (0 \leq j \leq N-1) \tag{25}$$

where $\lambda_j = -\kappa$ on T_{i1} and $\lambda_j = \lambda$ on T_{i2}. This implies

$$W(t_i + a_i) \leq W(t_i) \exp\{\frac{2C\Delta}{\varepsilon^3} [-\kappa(1-\varepsilon) + \lambda\varepsilon]\}. \tag{26}$$

Using $\xi^T = [x_p^T, x]$ and (20), (26) we finally obtain

$$\begin{aligned} \| x(t_i + a_i) \| &\leq \| \xi(t_i + a_i) \| \\ &\leq \| \xi(t_i) \| \cdot \left[\frac{\lambda_{max}(P)}{\lambda_{min}(p)} \right]^{1/2} \cdot \exp\{\frac{C\Delta}{\varepsilon^3} [-\kappa + \varepsilon(\lambda+\kappa)]\} \end{aligned}$$

$$\leq a_i[c_1 + c_2\varepsilon^3] \cdot \left[\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\right]^{1/2} \cdot$$

$$\cdot \exp\{\frac{C\Delta}{\varepsilon^3}[-\kappa + \varepsilon(\lambda+\kappa)]\}. \tag{27}$$

Obviously, the right hand side of (27) can be made less than a_i if ε is sufficiently small. But this is a contradiction to (22b). As a consequence, the assumption (21) was wrong, i.e. $x(t)$ is uniformly bounded, and so is $\xi(t)$ according to (20). □

5. Conclusions:

The model reference adaptive control problem in the presence of bounded disturbances is considered in this paper. The principal result of the paper is that if an upper bound $\|\theta^*\|_{max}$ on the norm of the unknown controller parameter vector θ^* is known and the adaptive law is suitably modified when $\|\theta\| \geq \|\theta^*\|_{max}$, then in spite of the disturbances all parameters and signals in the adaptive loop remain bounded. In the absence of disturbances the modification is of no significance and the output error $e(t)$ between the plant and the model goes to zero as t goes to infinity. This implies that the scheme suggested retains the properties of earlier schemes and is, in addition, robust with respect to bounded disturbances, which makes it suitable for use in practical applications.

As mentioned in the introduction, an alternate resolution of the problem is that if an upper bound on the magnitude of the disturbance is known, then a suitably designed deadzone in the adaptive law also guarantees the boundedness of all parameters and signals in the adaptive system [4]. In the latter approach the descent property of the parameter error vector $\phi(t)$ is always retained, whereas the output error $e(t)$ is only assured to lie within the deadzone as $t \to \infty$, even in the absence of disturbances. From this it is evident that the two approaches behave quite differently both in the presence of disturbances (in which $\dot{\phi}(t)$ may not go to zero in the scheme suggested in this paper) and in the absence of disturbances. Therefore, the analysis of their relative behavior deserves further investigation. Combining the two approaches appears to be possible and also interesting for practical applications.

In the adaptive law, which is considered in this paper, the parameter adjustment is based on the instantaneously available information. Instead, all the information that has become available up to time t could be used in order to improve the speed of adaptation properties, as suggested in [5]. The adaptive law then becomes

$$\dot{\theta}(t) = -\Gamma \int_0^t \frac{\omega(\tau)[\omega^T(\tau)\theta(t) - c_m^T \zeta_1(\tau) + y_p(\tau)]}{1 + x^T(\tau)x(\tau)} e^{-q(t-\tau)} dt - \Gamma\theta f(\theta) \tag{28}$$

where $q > 0$ is an arbitrary constant. The equation for the parameter error then is given by

$$\dot{\phi}(t) = -\Gamma \int_0^t \frac{\omega(\tau)[\omega^T(\tau)\phi(t) + \nu(\tau)]}{1 + x^T(\tau)x(\tau)} e^{-q(t-\tau)} dt - \Gamma\theta f(\theta) \tag{29}$$

Since (12) and (29) are structurally the same, the stability arguments of this paper can also be applied to show the stability for the adaptive law (29).

It is assumed in this paper that the high frequency gain k_p of the plant is known. This results in a relatively simple proof of stability. The basic idea of modifying the adaptive law based on knowledge of $\|\theta^*\|_{max}$ can, of course, also be applied when k_p is unknown. A proof of stability may then be obtained along the same lines as in this paper, although the details may be more involved.

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Appendix

(i) We first verify that there exists a positive constant $C_1 < \infty$ such that

$$|\nu| \leq C_1 \tag{A1}$$

$$\|\phi\| \leq C_1 \tag{A2}$$

$$\|\dot{x}/(1+x^T x)^{1/2}\| \leq C_1 \tag{A3}$$

$$\|\omega^{(i)}/(1+x^T x)^{1/2}\| \leq C_1 \ , \quad (0 \leq i \leq n^* + 1) \tag{A4}$$

$$\left| \frac{d}{dt} \{\phi^T \omega^{(i)}/(1+x^T x)^{1/2}\} \right| \leq C_1 \ , \quad (0 \leq i \leq n^*) \tag{A5}$$

where $\omega^{(i)}$ denotes the i-th time derivative of ω.

(A1) holds by assumption. (A2) was shown in section 3. (A3) follows from (A2) and (11). (A4) is true since $\zeta^{(i-1)}$ is proportional to ζ for $i = 1,\ldots,n^*$ and proportional to ζ and u_p, y_p for $i = n^* + 1$. From $u_p = v^T\theta + r'$ it follows that $u_p/(1+x^T x)^{1/2}$ is uniformly bounded because so are θ and r'. Since v is equal to the state of a nonminimal representation of the plant except for the bounded effect of the bounded disturbances ν_1, ν_2 it follows that $y_p/(1+x^T x)^{1/2}$ is also uniformly bounded. This establishes (A4). (A5) follows from (A4) and (A2),(12).

(ii) It is to be shown that

$$\left.\begin{aligned} &\|x(t)\| \geq 1/\varepsilon^3 && \text{(A6)} \\ &\frac{[\phi^T(t)\omega(t)]^2}{1 + x^T(t)x(t)} + \phi^T(t)\theta(t)f(\theta(t)) < \varepsilon && \text{(A7)} \end{aligned}\right\} \quad t \in [t_1, t_2]$$

and $t_2 - t_1 \geq 1$ implies

$$|\phi^T(t)v(t)|/[1 + x^T(t)x(t)]^{1/2} \leq h(\varepsilon) \qquad t \in [t_1, t_2] \tag{A8}$$

where $\{h: R^+ \to R^+\}$ is a continuous function and such that $\lim_{\varepsilon \to 0} h(\varepsilon) = 0$.

Since $\phi^T \theta f(\theta) \geq 0$, each term of the sum in (A7) is less than ε. Making use of the fact that $f(\theta) = 0$ for $\|\theta\| \leq \|\theta^*\|_{max}$, we obtain

$$f(\theta) < \frac{\varepsilon}{\phi^T \theta} = \frac{\varepsilon}{\theta^T\theta - \theta^T\theta^*}$$

$$\leq \frac{\varepsilon}{\|\theta\|} \cdot \frac{1}{\|\theta^*\|_{max} - \|\theta^*\|} \quad . \tag{A9}$$

(12) together with (A7),(A9) gives rise to

$$\|\dot{\phi}\| \leq \|\Gamma\| \cdot \left\{ \frac{\|\omega\|}{(1+x^Tx)^{1/2}} \cdot \frac{|\phi^T\omega| + |\nu|}{(1+x^Tx)^{1/2}} + \|\theta\| \cdot f(\theta) \right\}$$

$$\leq \|\Gamma\| \ \{ \sqrt{\varepsilon} + \varepsilon^3 C_1 + \varepsilon/(\|\theta^*\|_{max} - \|\theta^*\|)\}$$

$$\leq C_2 \sqrt{\varepsilon} \quad , \qquad (0 < \varepsilon \leq 1). \tag{A10}$$

Now let

$$|\phi^T\omega^{(i)}/(1+x^Tx)^{1/2}| \leq \varepsilon_i \quad , \qquad t \in [t_1,t_2] \tag{A11}$$

hold for some $\varepsilon_i > 0$ and $0 \leq i < n^*$. This is true for $i = 0$ and $\varepsilon_0 = \sqrt{\varepsilon}$ by the hypothesis (A7). For $i + 1$ we obtain the following identity by partial integration

$$\int_t^{t+\Delta t} \frac{\phi^T\omega^{(i+1)}}{(1+x^Tx)^{1/2}} dt = \left. \frac{\phi^T\omega^{(i)}}{(1+x^Tx)^{1/2}} \right|_t^{t+\Delta t} - \int_t^{t+\Delta t} \frac{\dot{\phi}^T\omega^{(i)}}{(1+x^Tx)^{1/2}} dt + \int_t^{t+\Delta t} \frac{\phi^T\omega^{(i)}}{(1+x^Tx)^{1/2}}$$

$$\cdot \frac{x^T\dot{x}}{1 + x^Tx} d\tau \quad . \tag{A12}$$

where $[t,t + \Delta t] \subset [t_1,t_2]$. Defining $C \triangleq \max\{1,C_1,C_2\}$ it follows that

$$\left| \int_t^{t+\Delta t} \frac{\phi^T\omega^{(i+1)}}{(1+x^Tx)^{1/2}} dt \right| \leq 2\varepsilon_i + \Delta t \cdot (C^2 \sqrt{\varepsilon} + \varepsilon_i C) \ . \tag{A13}$$

If the integrand on the left hand side of (A13) is equal to ε_{i+1} at time $t' \in [t_1,t_2]$, then by choosing $\Delta t = \varepsilon_{i+1}/C$ and t such that $t' \in [t,t+\Delta t]$, we obtain the result that the integral in (A13) is greater than or equal to $\Delta t \cdot \varepsilon_{i+1}/2 = \varepsilon_{i+1}^2/2C$. Inequality (A13) then can be rewritten as

$$\varepsilon_{i+1}^2 - 2\varepsilon_{i+1}(C^2\sqrt{\varepsilon} + \varepsilon_i C) - 4\varepsilon_i C \leq 0 \tag{A14}$$

Solving (A14) for the maximum possible ε_{i+1} we get

$$\varepsilon_{i+1} = (C^2\sqrt{\varepsilon} + \varepsilon_i C) + \{(C^2\sqrt{\varepsilon} + \varepsilon_i C)^2 + 4\varepsilon_i C\}^{1/2} . \tag{A15}$$

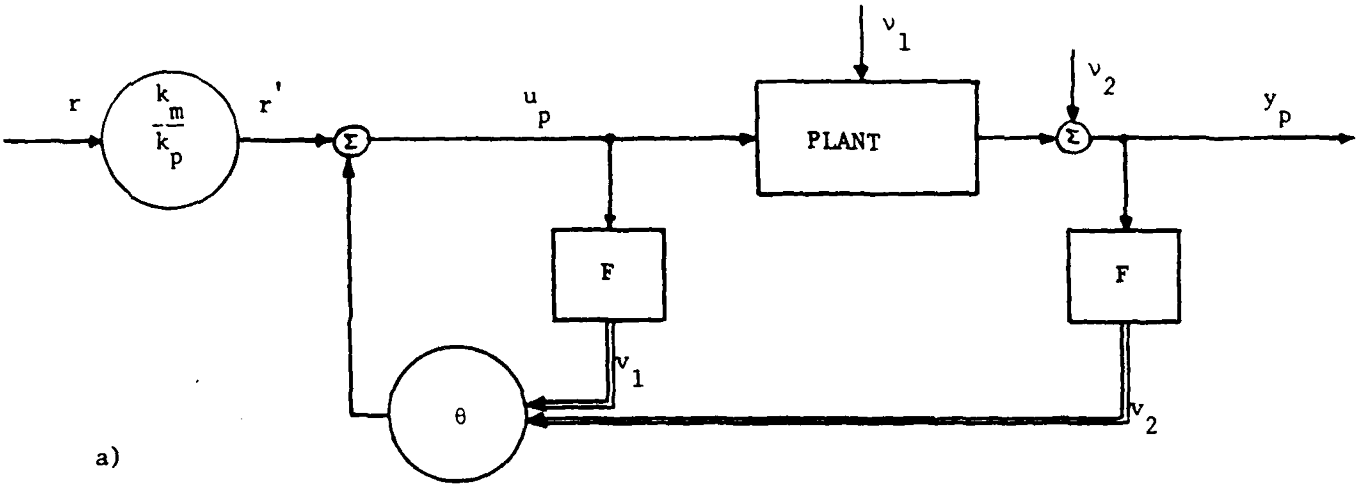
In conclusion, (A11) holds for $\varepsilon_0 = \sqrt{\varepsilon}$ and $\varepsilon_1,\ldots,\varepsilon_{n^*}$ being defined recursively by (A15).

Since $v = \sum_{i=0}^{n^*} d_i\omega^{(i)}$ from (3) and (5),(6), it follows that

$$\left| \frac{\phi^T v}{(1+x^T x)^{1/2}} \right| = \left| \sum_{i=0}^{n^*} d_i \frac{\phi^T \omega^{(i)}}{(1+x^T x)^{1/2}} \right|$$

$$\leq \sum_{i=0}^{n^*} d_i \varepsilon_i \triangleq h(\varepsilon) \tag{A16}$$

Hence $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$, which was the result to be proven.


r
$\frac{k_m}{k_p}$
r'
Σ
u_p
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Σ
y_p
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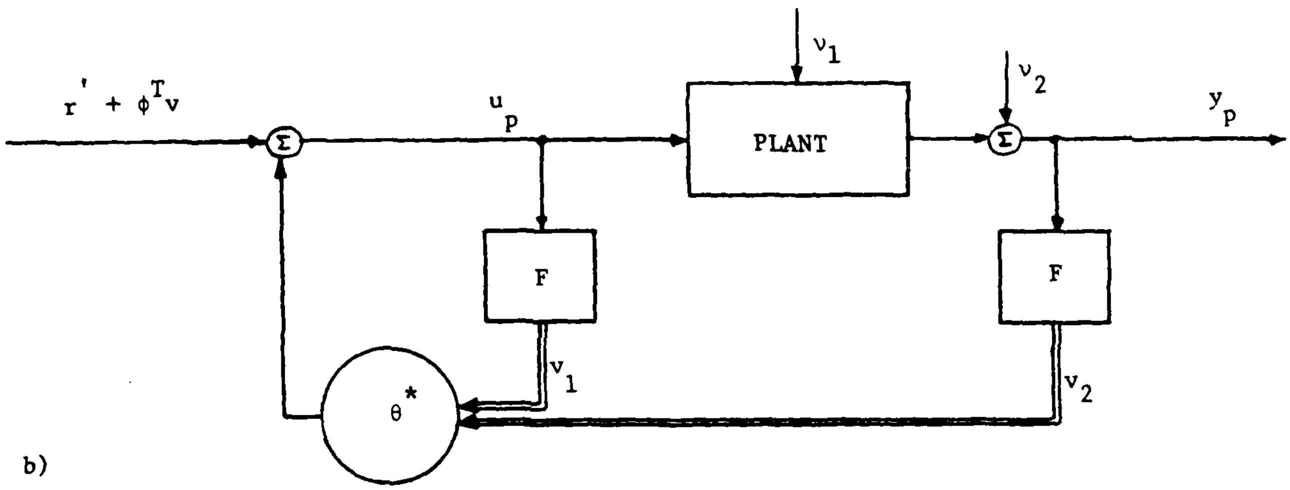
F
v_1
v_2
θ

a)


$r' + \phi^T v$
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v_1
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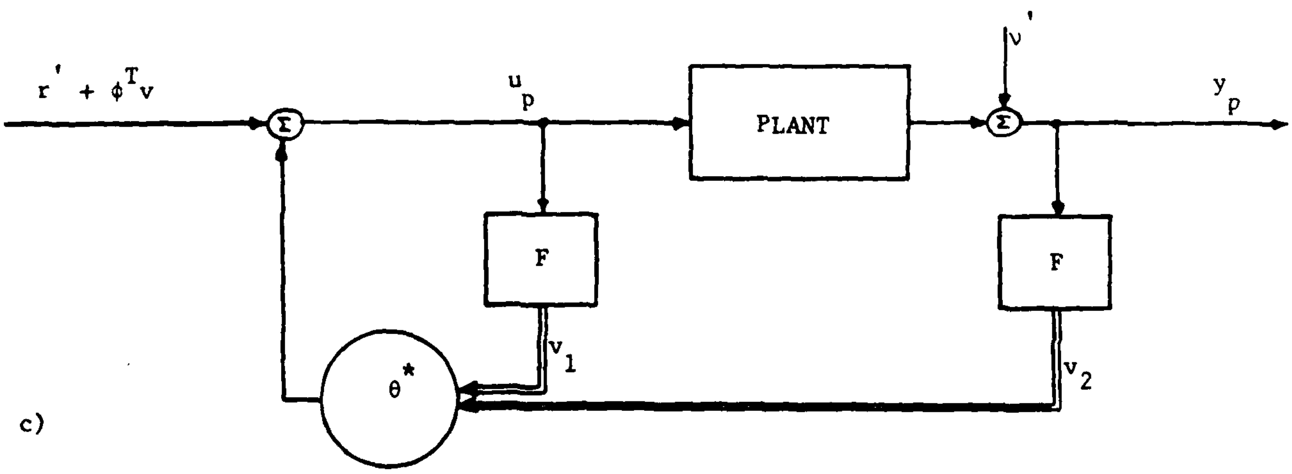
b)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c)

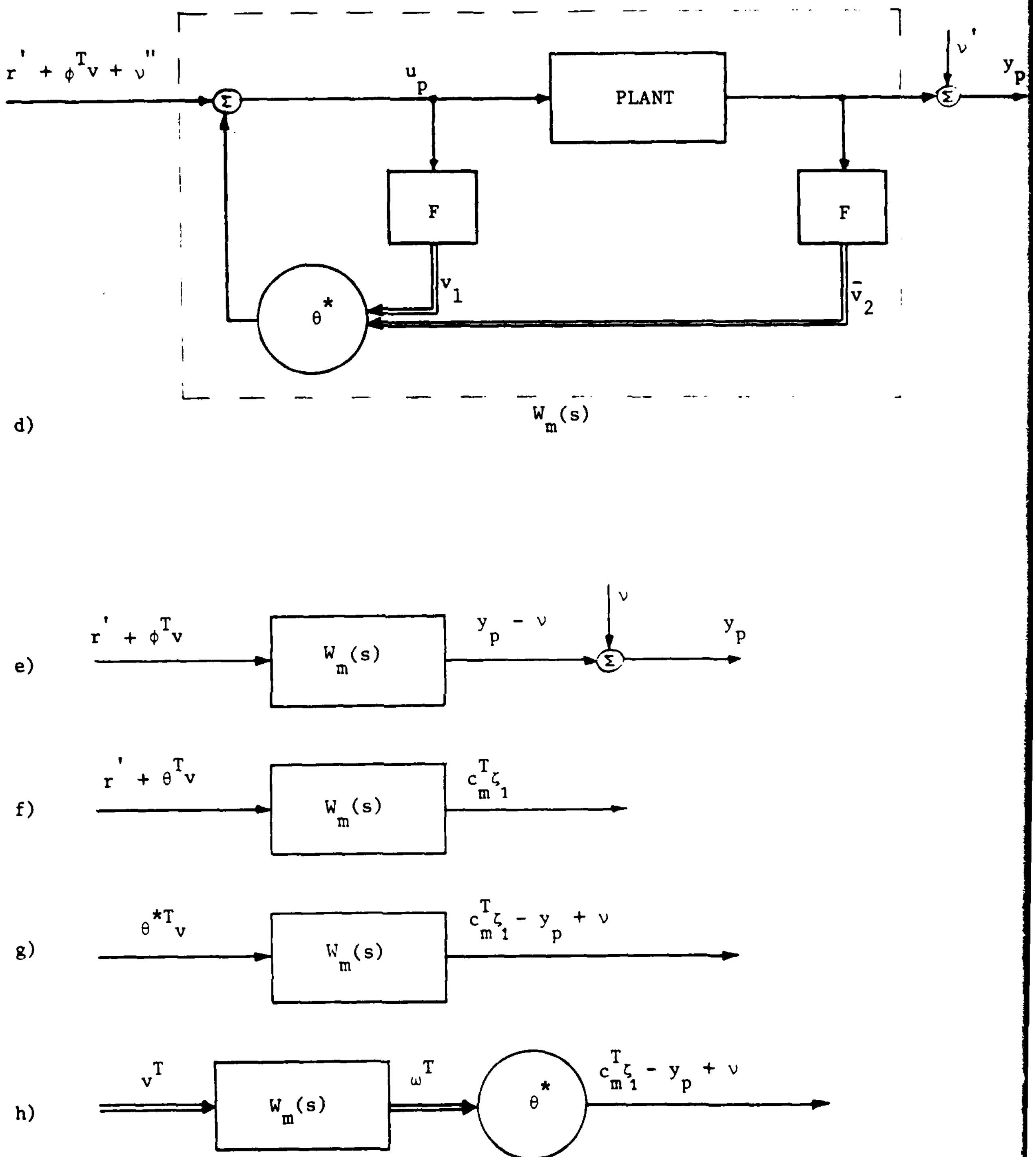

Fig. 1 Reformulation of the Adaptation Problem

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